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On trigonometric and paratrigonometric Hermite interpolation☆

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Abstract

In this paper, both trigonometric and paratrigonometric Hermite interpolation for any number of interpolation points with different multiplicities are constructed. The convergence of the Hermite trigonometric interpolation operator for 2π-periodic function and the Hermite paratrigonometric interpolation operator for 2π-antiperiodic function are given when the interpolated functions possess certain analyticity.

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1. Introduction

A good few discussions for Hermite trigonometric interpolation problem have been published. In [9], Salzer first discussed the Hermite trigonometric interpolation for non-

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equidistant interpolation points with uniform multiplicity by using flexibly the method of Chebyshev systems, but he has not given any analysis for the remainder term. Kress [7] established the Hermite trigonometric interpolation formula with equidistant interpolation points and uniform multiplicity. He introduced some ideas and tools important to the most succedent discussions on Hermite trigonometric interpolation and using them gave very explicit representations for both the fundamental Hermite polynomials and the remainder term. But we wonder that he avoided the singular integral of higher order in his process yielding the integral representations for the remainder term. In [1], Delvos considered, respectively, the π-periodic Hermite trigonometric interpolation and π-antiperiodic Hermite trigonometric interpolation for any odd and any even number of interpolation points with different multiplicities. His approach might be considered as an extension of the method of Salzer and a generalization of Kress's idea to non-equidistant interpolation points with different multiplicities. He has also not given any analysis of the remainder term and those trigonometric interpolation polynomials obtained are not ones of minimum degree in the family of all trigonometric polynomials. In other words, they are the Hermite trigonometric interpolation polynomials with the understanding under the scheme of the π-translation nodes. In 1994, Dryanov [2] proved the existence and uniqueness of the Hermite trigonometric interpolation polynomial for general case, any number of interpolation points and any multiplicities by the method based on Chebyshev systems. Using this method to get the constructive fundamental Hermite trigonometric polynomials is very difficult. In 1997, Jin [3] established constructively the fundamental Hermite polynomials for the general case. He refered to [1,4] and [7] on Lagrange trigonometric interpolation. Until quite recently the research on Hermite paratrigonometric interpolation problem has been completely ignored, although it can yield excellent results for the quadrature formulas of singular integral with the cosecant kernel, which will be given in a forthcoming paper. In addition, we point out an interesting fact that Hermite trigonometric interpolation problem and paratrigonometric interpolation problem are twins and their solutions are completely parallel. In the present paper, we shall slightly modify the generating function used to Taylor trigonometric interpolation in [1,7] due to Delvos and Kress. Then, the general Hermite paratrigonometric interpolation and trigonometric interpolation are constructively given as well. If the interpolated function has certain analyticity, a clear integral representation for the remainder term will also be obtained by the residue theorem of singular integrals of higher order due to Jian-ke Lu [8].

2. Trigonometric and paratrigonometric polynomials

We use $C_{2\pi}$ to denote the family of all 2π-periodic continuous functions. Let H_n^T denote the class of all trigonometric polynomials of degree not greater than n and regard $H_n^T = \{0\}$ if $n < 0$. Let $H_n^T(\alpha)$ denote the family of trigonometric polynomials of the form

$$a_n \sin(nt + \alpha) + T_{n-1}(t), \quad T_{n-1} \in H_{n-1}^T, \quad 0 \leqslant \alpha < \pi, \; n > 1 \tag{2.1}$$

and regard $H_n^T(\alpha) = \{0\}$ if $n < 0$, $H_0^T(0) = \{0\}$ and $H_0^T(\frac{\pi}{2}) = \{all \;\; constants\}$. In (2.1) a_n is called the coefficient of term of degree n. It is obvious that any trigonometric polynomial of degree n $(n \geqslant 0)$ cannot belong to two different classes $H_n^T(\alpha_1)$ and $H_n^T(\alpha_2)$ $(\alpha_1 \neq \alpha_2)$.

If f is continuous and $f(t+2\pi) = -f(t)$ then we call it to be 2π-antiperiodic and write it by $f \in \overline{C}_{2\pi}$. If

$$T_{n+\frac{1}{2}}(t) = \sum_{j=0}^{n} [a_j \sin(j+\tfrac{1}{2})t + b_j \cos(j+\tfrac{1}{2})t] \quad \text{with} \quad a_n^2 + b_n^2 \neq 0, \tag{2.2}$$

then we call it a paratrigonometric polynomial of degree $n+\frac{1}{2}$. Let $H^T_{n+\frac{1}{2}}$ denote the class of all paratrigonometric polynomials of degree not greater than $n+\frac{1}{2}$. Obviously

$$H^T_{n+\frac{1}{2}} \subset \overline{C}_{2\pi}, \tag{2.3}$$

so $T_{n+\frac{1}{2}}$ is also called a trigonometric polynomial of degree $n+\frac{1}{2}$ in $\overline{C}_{2\pi}$. We again regard $H^T_{n+\frac{1}{2}} = \{0\}$ if $n<0$. Let $H^T_{n+\frac{1}{2}}(\alpha)$ denote the family of all paratrigonometric polynomials of the form

$$a_n \sin[(n+\tfrac{1}{2})t + \alpha] + T_{n-\frac{1}{2}}(t), \quad T_{n-\frac{1}{2}} \in H^T_{n-\frac{1}{2}}, \quad 0 \leqslant \alpha < \pi \tag{2.4}$$

and regard $H^T_{n+\frac{1}{2}}(\alpha) = \{0\}$ if $n<0$. a_n is called the coefficient of the term of degree $n+\frac{1}{2}$. It is obvious that any paratrigonometric polynomial of degree $n+\frac{1}{2}$ cannot belong to two different classes $H^T_{n+\frac{1}{2}}(\alpha_1)$ and $H^T_{n+\frac{1}{2}}(\alpha_2)$ $(\alpha_1 \neq \alpha_2)$.

Lemma 2.1 (*Du Jinyuan and Liu Hua*[6]). *If $F \in H^T_{\frac{1}{2}k}(\alpha)$ is of degree $\frac{1}{2}k$ $(k \geqslant 0)$, $G \in H^T_{\frac{1}{2}r}(\beta)$ is of degree $\frac{1}{2}r$ $(r \geqslant 0)$, then $FG \in H^T_{\frac{1}{2}(k+r)}([\frac{1}{2}\pi + \alpha + \beta]_\pi)$ is of degree $\frac{1}{2}(k+r)$, where $[\theta]_\pi$ denotes the number congruent to θ* $(\mathrm{mod}\,\pi)$ in $[0,\pi)$.

Proof. It is enough to prove the case when $F(t) = \sin(\frac{1}{2}kt + \alpha)$ and $G(t) = \sin(\frac{1}{2}rt + \beta)$. This follows from the relation $F(t)G(t) = \frac{1}{2}\{\cos[\frac{1}{2}(k-r)t + \alpha - \beta] - \sin[\frac{1}{2}(k+r)t + \frac{1}{2}\pi + \alpha + \beta]\}$. $\square$

Corollary 2.1. $\sin^{m+1}\frac{1}{2}t \cot \frac{1}{2}t \in H^T_{\frac{1}{2}(m+1)}\left(\left[\frac{1}{2}(m+1)\pi\right]_\pi\right)$ *for* $m = 0, 1, \ldots$.

Lemma 2.2. *If $F \in H^T_{\frac{1}{2}k}(\alpha)$ is of degree $\frac{1}{2}k$ $(k \geqslant 0)$, then $F' \in H^T_{\frac{1}{2}k}([\frac{\pi}{2} + \alpha]_\pi)$ and is of degree $\frac{1}{2}k$.*

Lemma 2.3. *If $F \in H^T_{\frac{1}{2}k}$ is of degree $\frac{1}{2}k$ $(k \geqslant 0)$, then it has at most k zero points on* $[0, 2\pi)$.

Proof. Since F^2 is a trigonometric polynomial of k, so it has at most $2k$ zero points on $[0,2\pi)$, thus, F has at most k zero points on $[0,2\pi)$.

Define the function

$$K_m(t) = \sin^{m+1}\tfrac{1}{2}t\, D^m \cot\tfrac{1}{2}t \quad (m = 1, 2, \ldots) \quad \text{with} \quad D := \frac{\mathrm{d}}{\mathrm{d}t}, \tag{2.5}$$

which is a minor modification of the generating function in [1,7]. It is clear that

$$K_1(t) = -\tfrac{1}{2} \in H_0^T\left(\tfrac{1}{2}\pi\right), \quad K_2(t) = \tfrac{1}{2}\cos\tfrac{1}{2}t \in H_{\frac{1}{2}}^T\left(\tfrac{1}{2}\pi\right). \quad \square \tag{2.6}$$

In general, Kress has given the following results [7].

Lemma 2.4. $K_m \in H^T_{\frac{1}{2}(m-1)}\left(\frac{1}{2}\pi\right)$.

In fact,

$$\begin{aligned} K_{m+1}(t) &= \sin^{m+2}\tfrac{1}{2}t\, D^{m+1}\cot\tfrac{1}{2}t \\ &= D\left(\sin^{m+2}\tfrac{1}{2}t\, D^m \cot\tfrac{1}{2}t\right) - \left(D\sin^{m+2}\tfrac{1}{2}t\right) D^m\cot\tfrac{1}{2}t \\ &= \sin\tfrac{1}{2}t\, DK_m(t) - \tfrac{m+1}{2}\cos\tfrac{1}{2}t\, K_m(t), \end{aligned}$$

using Lemmas 2.1 and 2.2, the present lemma is established by the inductive method.

Corollary 2.2. *If* $p \geqslant m \geqslant 1$, *then* $\sin^{p+1}\frac{1}{2}t\, D^m\cot\frac{1}{2}t \in H^T_{\frac{1}{2}(p-1)}\left([\frac{1}{2}(p+1-m)\pi]_\pi\right)$.

3. Trigonometric and paratrigonometric Hermite interpolation

Let $t_1 < t_2 < \cdots < t_n$ be n distinct points in $[0, 2\pi)$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n positive integers, $\{d_{j,\ell}\}_{1,\ 0}^{n,\ \lambda_j-1} = \{d_{j,\ell},\ j = 1, 2, \ldots, n,\ \ell = 0, 1, \ldots, \lambda_j - 1\}$ be a set of $\lambda = \sum_{j=1}^{n} \lambda_j$ given numbers.

Problem THI. Find a trigonometric polynomial T of minimum degree which satisfies the following interpolation conditions

$$D^\ell\, T(t_j) = d_{j,\ell}, \quad j = 1, 2, \ldots, n, \quad \ell = 0, 1, \ldots, \lambda_j - 1. \tag{3.1}$$

Problem PTHI. Find a paratrigonometric polynomial T of minimum degree such that it satisfies the interpolation conditions in (3.1).

The first problem is well known as the Hermite trigonometric interpolation problem, the second problem is the Hermite paratrigonometric interpolation problem. Thereinafter we shall see that their solutions are completely alternative and parallel. So, in either event we clearly write T by $T\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}\{d_{j,\ell}\}_{1,\ 0}^{n,\ \lambda_j-1}$.

The points $t_1, t_2, \ldots, t_n$ are called the nodes and λ_j is called the multiplicity of the node t_j.

Firstly, we do the simplest Taylor interpolation problem: $n = 1$, $t_1 = 0$ and $\lambda = \lambda_1$. To do so, recall that the sinc function is defined by $\operatorname{sinc} x = \sin x/x$ which is an even entire function satisfying $\operatorname{sinc}(0) = 1$. Then the reciprocal function $1/\operatorname{sinc}(x)$ is analytic in $|x| < \pi$ and possesses the Taylor series

$$\frac{x}{\sin\ x} = \sum_{j=0}^{+\infty} a_{2j} x^{2j}.$$

Now we have to consider the analytic function

$$\left(\frac{x}{2\ \sin\ \frac{x}{2}}\right)^{m+1} = \sum_{j=0}^{+\infty} a_{2j}(m) x^{2j}, \quad |x| < 2\pi, \tag{3.2}$$

where m is non-negative integer.

Lemma 3.1. *Let m and s be integers with $0 \leqslant s \leqslant m$. Then*

$$h_{s,m}(t) = \sin^{m+1} \tfrac{1}{2}t\ \frac{(-1)^{m-s} 2^m}{s!} \sum_{\ell=0}^{[\frac{m-s}{2}]} \frac{a_{2\ell}(m)}{(m-s-2\ell)!}$$
$$\times D^{m-s-2\ell} \cot\ \tfrac{1}{2}t \in H^T_{\frac{1}{2}(m+1)}\left([\tfrac{1}{2}(m+1)\pi]_\pi\right) \tag{3.3}$$

and

$$D^\ell h_{s,m}(0) = \delta_{\ell,s}, \quad 0 \leqslant \ell, s \leqslant m, \tag{3.4}$$

where $[x]$ denotes the integer part of x, $a_{2\ell}(m)$ is the coefficient of the Taylor series (3.2), *$\delta_{\ell,s}$ is Kronecker's symbol*.

Proof. It is easy to prove that $h_{s,m} \in H^T_{\frac{1}{2}(m+1)}\left([\frac{1}{2}(m+1)\pi]_\pi\right)$ by Corollaries 2.1 and 2.2. Noting that, in some neighborhood of $z = 0$,

$$D^q \cot\ \tfrac{1}{2}z = (-1)^q \frac{2q!}{z^{q+1}} + D^q H(z), \quad q = 0, 1, \ldots, \tag{3.5}$$

where the function $H(z)$ is analytic, we obtain

$$\begin{aligned} h_{s,m}(t) =& \sin^{m+1}\tfrac{1}{2}t\,\frac{(-1)^{m-s}2^m}{s!}\sum_{\ell=0}^{[\frac{m-s}{2}]}\frac{a_{2\ell}(m)}{(m-s-2\ell)!} \\ &\times\left[(-1)^{m-s}\frac{2(m-s-2\ell)!}{t^{m-s-2\ell+1}}+D^{m-s-2\ell}H(t)\right] \\ =& \operatorname{sinc}^{m+1}\tfrac{1}{2}t\,\frac{t^s}{s!}\sum_{\ell=0}^{[\frac{m-s}{2}]}a_{2\ell}(m)t^{2\ell} \\ &+\sin^{m+1}\tfrac{1}{2}t\,\frac{(-1)^{m-s}2^m}{s!}\sum_{\ell=0}^{[\frac{m-s}{2}]}\frac{a_{2\ell}(m)}{(m-s-2\ell)!}D^{m-s-2\ell}H(t) \\ =& \frac{t^s}{s!}-\sin^{m+1}\tfrac{1}{2}t\,H_{s,m}(t), \end{aligned}$$

where

$$\begin{aligned} H_{s,m}(t) =& \left(\frac{2}{t}\right)^{m+1}\frac{t^s}{s!}\sum_{\ell=[\frac{m-s}{2}]+1}^{\infty}a_{2\ell}(m)t^{2\ell}-\frac{(-1)^{m-s}2^m}{s!}\sum_{\ell=0}^{[\frac{m-s}{2}]} \\ &\times\frac{a_{2\ell}(m)}{(m-s-2\ell)!}D^{m-s-2\ell}H(t). \end{aligned}$$

By $2\ell+s-m-1\geqslant 0$ in the case $\ell\geqslant[\frac{m-s}{2}]+1$, the function $H_{s,m}(t)$ is analytic. Then, we get

$$D^\ell h_{s,m}(0)=\delta_{\ell,s},\quad 0\leqslant\ell,s\leqslant m. \qquad \Box$$

Remark 3.1. In fact, $a_0(m)=1$, so

$$h_{m-1,m}(t)=-\sin^{m+1}\tfrac{1}{2}t\,\frac{2^m}{(m-1)!}a_0(m)\,D\cot\tfrac{1}{2}t=\frac{2^{m-1}}{(m-1)!}\sin^{m-1}\tfrac{1}{2}t.$$

By Lemma 2.1, $h_{m-1,m}\in H^T_{\frac{1}{2}(m-1)}\left([\tfrac{1}{2}m\pi]_\pi\right)$ is of degree $\frac{1}{2}(m-1)$.

Let

$$\omega_{s,\lambda}(t)=\begin{cases} h_{s,\lambda-2}(t)-D^{\lambda-1}h_{s,\lambda-2}(0)h_{\lambda-1,\lambda}(t), & s=0,1,\ldots,\lambda-2,\\ h_{\lambda-1,\lambda}(t), & s=\lambda-1.\end{cases} \tag{3.6}$$

By Lemma 3.1 and Remark 3.1, we get

$$\begin{cases}\omega_{s,\lambda}\in H^T_{\frac{1}{2}(\lambda-1)}, & s=0,1,\ldots,\lambda-1,\\ D^\ell\omega_{s,\lambda}(0)=\delta_{\ell,s}, & \ell=0,1,\ldots,\lambda-1.\end{cases} \tag{3.7}$$

In other words, we have the following result.

Lemma 3.2. *When λ is even, $\omega_{s,\lambda}$ $(s=0, 1, \ldots, \lambda-1)$ are the fundamental paratrigonometric polynomials for Taylor paratrigonometric interpolation problem at the point* 0. *When λ is odd, $\omega_{s,\lambda}$ $(s=0, 1, \ldots, \lambda–1)$ are the fundamental trigonometric polynomials for Taylor trigonometric interpolation problem at* 0.

Nextly, we are going to construct the fundamental functions for Problems THI and PTHI, i.e., while $d_{r,k} = 1$ for certain (r, k) and the others vanish, we find $T_{r,k}$ such that

$$D^{\ell}\, T_{r,k}(t_j) = \delta_{j,r}\delta_{\ell,k}, \quad j = 1, 2, \ldots, n, \;\; \ell = 0, 1, \ldots, \lambda_j - 1. \tag{3.8}$$

To do this, let

$$\Delta_n(t) = \prod_{j=1}^{n} \sin^{\lambda_j} \frac{1}{2}(t - t_j), \tag{3.9}$$

$$\Delta_{n,r}(t) = \prod_{j=1, j\neq r}^{n} \sin^{\lambda_j} \frac{1}{2}(t - t_j). \tag{3.10}$$

Noting that $\Delta_{n,r}(t_r) \neq 0$, we introduce the following function:

$$\phi_{r,k}(t) = \frac{(t - t_r)^k}{k!\Delta_{n,r}(t)} \tag{3.11}$$

and write

$$\phi^{\ell}_{r,k} = D^{\ell}\phi_{r,k}(t_r), \quad \ell = 0, 1, \ldots, \lambda_r - 1 \tag{3.12}$$

In fact,

$$\phi^{\ell}_{r,k} = 0 \;\; (\ell < k), \quad \phi^{k}_{r,k} = \frac{1}{\Delta_{n,r}(t_r)}. \tag{3.13}$$

Obviously

$$\sum_{\ell=k}^{s} C^{\ell}_s D^{s-\ell}\Delta_{n,r}(t_r)\phi^{\ell}_{r.k} = \sum_{\ell=0}^{s} C^{\ell}_s D^{s-\ell}\Delta_{n,r}(t_r)\phi^{\ell}_{r.k} = \delta_{s,k}, \quad s = 0, 1, \ldots, \lambda_r - 1. \tag{3.14}$$

Introduce the function

$$\theta_{r,k}(t) = \sum_{s=k}^{\lambda_r - 1} \phi^{s}_{r,k}\omega_{s,\lambda_r}(t), \tag{3.15}$$

where ω_{s,λ_r}'s are given by (3.6) with $\lambda = \lambda_r$.

Remark 3.2. By Remark 3.1 and (3.13), we know that $\theta_{r,\lambda_r-1}(t) = \frac{2^{\lambda_r-1}}{(\lambda_r-1)!\Delta_{n,r}(t_r)} \sin^{\lambda_r-1} \frac{1}{2}t$. In particular, if $\lambda_r = 1$ then $\theta_{r,0}(t) = \frac{1}{\Delta_{n,r}(t_r)}$.

By (3.7), (3.15), Lemma 2.3 and (3.13), we may get

$$\begin{cases} \theta_{r,k} \in H^T_{\frac{1}{2}(\lambda_r-1)} & \text{with order} \geqslant \frac{1}{2}k, \\ D^s\theta_{r,k}(0) = \phi^s_{r,k}, & s = 0, 1, \ldots, \lambda_r - 1. \end{cases} \tag{3.16}$$

Let

$$T_{r,k}(t) = \varDelta_{n,r}(t)\theta_{r,k}(t - t_r), \tag{3.17}$$

then, by (3.10), (3.14) and (3.16), we easily see

$$\begin{cases} T_{r,k} \in H^T_{\frac{1}{2}(\lambda-1)}, & \lambda = \sum_{j=1}^{n} \lambda_j, \\ D^s T_{r,k}(t_j) = \delta_{j,r}\delta_{s,k}, & j = 1, 2, \ldots, n, \quad s = 0, 1, \ldots, \lambda_j - 1. \end{cases} \tag{3.18}$$

To sum up, we get

$$\begin{aligned} T_{r,k}(t) = & \frac{2^{\lambda_r-1}\varDelta_n(t)}{\sin\frac{1}{2}(t-t_r)}\left\{\frac{1}{(\lambda_r-1)!}\left(\phi^{\lambda_r-1}_{r,k} - \sum_{\ell=k}^{\lambda_r-2}\phi^{\ell}_{r,k}D^{\lambda_r-1}h_{\ell,\lambda_r-2}(0)\right)\right. \\ & +\sum_{s=k}^{\lambda_r-2}\frac{(-1)^{\lambda_r-s}\phi^s_{r,k}}{2s!}\sum_{\ell=0}^{\left[\frac{1}{2}(\lambda_r-s)\right]-1} \\ & \left.\times\frac{a_{2\ell}(\lambda_r-2)}{(\lambda_r-2-s-2\ell)!}D^{\lambda_r-2-s-2\ell}\cot\frac{1}{2}(t-t_r)\right\}, \end{aligned} \tag{3.19}$$

where

$$\phi^s_{r,k} = D^s\left[\frac{(t-t_r)^k}{k!\varDelta_{n,r}(t)}\right]_{t=t_r} \tag{3.20}$$

and the above sum $\sum_{\ell=k}^{\lambda_r-2}$ vanishes if $k = \lambda_r - 1$.

Remark 3.3. As a special example, we point out

$$\begin{cases} T_{r,\lambda_r-1}(t) = \frac{2^{\lambda_r-1}\varDelta_{n,r}(t)}{(\lambda_r-1)!\varDelta_{n,r}(t_r)}\sin^{\lambda_r-1}\frac{1}{2}(t-t_r) \\ \qquad = \frac{\lambda_r\varDelta_n(t)}{2D^{\lambda_r}\varDelta_n(t_r)}\csc\frac{1}{2}(t-t_r), \\ T_{r,0}(t) = \frac{\varDelta_{n,r}(t)}{\varDelta_{n,r}(t_r)} = \frac{\varDelta_n(t)}{2\varDelta'_n(t_r)}\csc\frac{1}{2}(t-t_r) \quad \text{while } \lambda_r = 1. \end{cases} \tag{3.21}$$

Theorem 3.1. *Let*

$$T(t) = T\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}\{d_{r,k}\}_{1,\ \ 0}^{n,\ \lambda_j-1}(t) = \sum_{r=1}^{n}\sum_{k=0}^{\lambda_r-1} d_{r,k}T_{r,k}(t), \tag{3.22}$$

where $T_{r,k}$ is given by (3.19) *with* (3.20). *Then, $T \in H^T_{\frac{1}{2}(\lambda-1)}$ is the unique solution of Problem PTHI when $\lambda = \sum_{j=1}^{n} \lambda_j$ is even, $T \in H^T_{\frac{1}{2}(\lambda-1)}$ is the unique solution of Problem THI when $\lambda = \sum_{j=1}^{n} \lambda_j$ is odd.*

Proof. By (3.18), $T \in H^T_{\frac{1}{2}(\lambda-1)}$. So, it is a paratrigonometric polynomial and trigonometric polynomial, respectively, when λ is even and odd. Obviously, the interpolation condition (3.1) is satisfied for T. Now we prove that it has minimum degree. To do this, it is enough to prove the uniqueness in $H^T_{\frac{1}{2}(\lambda-1)}$ for T. In fact, if $G = G\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}\{d_{r,k}\}_{1,\ \ 0}^{n,\ \lambda_j-1}$ is another one, then $D^\ell T(t_j) - D^\ell G(t_j) = 0$ $(j = 1, 2, \ldots, n,\ \ell = 0, 1, \ldots, \lambda_j - 1)$, Thus $T = G$ by Lemma 2.3. □

Example 3.1. If $T \in H^T_{\frac{1}{2}(\lambda-1)}$ with odd λ and $G(t) = T(2t)$, then $G \in H^T_{\lambda-1}$ and $G(\pi + t) = G(t)$, in this case G is said to be π-periodic [1]. In reverse, if $G \in H^T_{\lambda-1}$ is π-periodic, then we easily prove $G(t) = \sum_{j=\text{even}}^{\lambda} [a_j \sin jt + b_j \cos jt]$, so $T \in H^T_{\frac{1}{2}(\lambda-1)}$. Similarly, $G \in H^T_{\lambda-1}$ and $G(\pi+t) = -G(t)$ (G is said to be π-antiperiodic [1]) is equivalent to $T \in H^T_{\frac{1}{2}(\lambda-1)}$ with even λ. Let $x_j = \frac{1}{2}t_j$ $(j = 1, 2, \ldots, n)$ and $c_{j,k} = 2^k d_{j,k}$ $(j = 1, 2, \ldots, n,\ k = 0, 1, \ldots, \lambda_j - 1)$, then, for $j = 1, 2, \ldots, n$, $k = 0, 1, \ldots, \lambda_j - 1$,

$$D^k G(x_j) = c_{j,k} \Longleftrightarrow D^k T(t_j) = d_{j,k}.$$

In this way, we may see that both the π-periodic $G \in H^T_{\lambda-1}$ satisfying $D^k G(x_j) = c_{j,k}$ when λ is odd and the π-antiperiodic $G \in H^T_{\lambda-1}$ satisfying $D^k G(x_j) = c_{j,k}$ when λ is even are $T\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}\{d_{r,k}\}_{1,\ \ 0}^{n,\ \lambda_j-1}$. Thus they uniquely exist, which is just the result in [1].

The remainder of Problems PTHI and THI is slightly complex. Firstly, if we find the solution in $H^T_{\frac{1}{2}\lambda}$, then both Problem PTHI with odd λ and Problem THI with even λ are not properly posed. Since

$$\Delta_n \in H^T_{\frac{1}{2}\lambda}(\phi) \tag{3.23}$$

where

$$\phi=\begin{cases}\left[-\frac{1}{2}\sum_{j=1}^{n}\lambda_j t_j\right]_\pi & \text{if } \lambda \text{ is odd,}\\ \left[\frac{1}{2}\pi-\frac{1}{2}\sum_{j=1}^{n}\lambda_j t_j\right]_\pi & \text{if } \lambda \text{ is even,}\end{cases} \tag{3.24}$$

then, $T\in H^T_{\frac{1}{2}\lambda}$ is a solution, so does $T+\Delta_n$. Secondly, if we find the solution in $H^T_{\frac{1}{2}\lambda-1}$, then both Problem PTHI with odd λ and Problem THI with even λ are pathological. In fact, if $T\in H^T_{\frac{1}{2}\lambda-1}$ and satisfies the interpolation conditions $D^\ell T(t_j)=\delta_{\ell,\lambda_n-1}(j=1,2,\ldots,n,\ \ell=0,1,\ldots,\lambda_j-1)$, then $T=0$ by Lemma 2.3, but $D^{\lambda_n-1}T(t_n)=1$. Of course, if we find the solution $T\in H^T_{\frac{1}{2}\lambda}(\phi)$, then there exists constant C such that $T-C\Delta\in H^T_{\frac{1}{2}\lambda-1}$, thus Problems PTHI and THI again become as finding the solution in the space $H^T_{\frac{1}{2}\lambda-1}$. In words, to find the solution in $H^T_{\frac{1}{2}\lambda}(\alpha)$ $(\alpha\neq\phi)$ is appropriate.

Now, we come to construct the fundamental Hermite functions T_{r,λ_r-1} $(r=1,2,\ldots,n)$ in $H^T_{\frac{1}{2}\lambda}(\alpha)$ as follows:

Let

$$\Delta^\alpha_{n,r}(t)=\Delta_{n,r}(t)\sin^{\lambda_r-1}\frac{1}{2}(t-t_r)\sin\left(\frac{1}{2}(t-t_r)+\alpha-\phi\right)\quad(\alpha\neq\phi). \tag{3.25}$$

Then, by Lemma 2.1 and noting $\alpha\neq\phi$, we get

$$\begin{cases}\Delta^\alpha_{n,r}\in H^T_{\frac{1}{2}\lambda}(\alpha) \quad \text{with degree } \frac{1}{2}\lambda,\\ D^{\lambda_r-1}\Delta^\alpha_{n,r}(t_r)=2^{1-\lambda_r}(\lambda_r-1)!\Delta_{n,r}(t_r)\sin(\alpha-\phi)\neq 0.\end{cases} \tag{3.26}$$

Set up

$$\begin{aligned}T_{r,\lambda_r-1}(t)&=\frac{\Delta^\alpha_{n,r}(t)}{D^{\lambda_r-1}\Delta^\alpha_{n,r}(t_r)}\\ &=\frac{\lambda_r\Delta_n(t)\sin\left(\frac{1}{2}(t-t_r)+\alpha-\phi\right)}{2D^{\lambda_r}\Delta_n(t_r)\sin(\alpha-\phi)}\csc\frac{1}{2}(t-t_r),\end{aligned} \tag{3.27}$$

obviously

$$\begin{cases}T_{r,\lambda_r-1}\in H^T_{\frac{1}{2}\lambda}(\alpha)\\ \quad\text{with order } \frac{1}{2}\lambda, \qquad r=1,2,\ldots,n,\\ D^\ell T_{r,\lambda_r-1}(t_j)=\delta_{j,r}\delta_{\ell,\lambda_r-1}, \quad j=1,2,\ldots,n,\ \ell=0,1,\ldots,\lambda_j-1.\end{cases} \tag{3.28}$$

Remark 3.4. Taking $[\alpha-\phi]_\pi=\frac{1}{2}\pi$, then

$$T_{r,\lambda_r-1}(t)=\frac{\lambda_r\Delta_n(t)}{2D^{\lambda_r}\Delta_n(t_r)}\cot\frac{1}{2}(t-t_r) \tag{3.29}$$

and

$$T_{r,0}(t) = \frac{\Delta_n(t)}{2\Delta_n'(t_r)} \cot \frac{1}{2}(t - t_r) \tag{3.30}$$

while $\lambda_r = 1$.

Next, we construct the fundamental Hermite functions $T_{r,k}$ $(r = 1, 2, \ldots, n,\ k = 0, 1, \ldots, \lambda_r - 2)$ $(\lambda_r \geqslant 2)$ in $H^T_{\frac{1}{2}\lambda}(\alpha)$ as follows:

By using Theorem 3.1, there exist $\Upsilon_{r,k}$ such that

$$\begin{cases} \Upsilon_{r,k} \in H^T_{\frac{1}{2}\lambda-1}, & r = 1, 2, \ldots, n,\ k = 0, 1, \ldots, \lambda_r - 2, \\ D^\ell \Upsilon_{r,k}(t_j) = \delta_{j,r}\delta_{\ell,k}, & j = 1, 2, \ldots, n,\ \ell = 0, 1, \ldots, \lambda_j - 1 - \delta_{j,r}. \end{cases} \tag{3.31}$$

Let

$$T_{r,k}(t) = \Upsilon_{r,k}(t) - D^{\lambda_r-1}\Upsilon_{r,k}(t_r)T_{r,\lambda_r-1}(t), \tag{3.32}$$

then, by (3.31) and (3.28), we get

$$\begin{cases} T_{r,k} \in H^T_{\frac{1}{2}\lambda}(\alpha), & r = 1, 2, \ldots, n,\ k = 0, 1, \ldots, \lambda_r - 2, \\ D^\ell T_{r,k}(t_j) = \delta_{j,r}\delta_{\ell,k}, & j = 1, 2, \ldots, n,\ \ell = 0, 1, \ldots, \lambda_r - 1. \end{cases} \tag{3.33}$$

To sum up the above results, we get

$$\begin{cases} T_{r,\lambda_r-1}(t) = \dfrac{\lambda_r \Delta_n(t)\, \sin\left(\frac{1}{2}(t-t_r)+\alpha-\phi\right)}{2D^{\lambda_r}\Delta_n(t_r)\sin(\alpha-\phi)} \csc \frac{1}{2}(t - t_r), \\ \\ T_{r,k}(t) = \Upsilon_{r,k}(t) - D^{\lambda_r-1}\Upsilon_{r,k}(t_r)T_{r,\lambda_r-1}(t), \quad k = 0, 1, \ldots, \lambda_r - 2 \end{cases} \tag{3.34}$$

where

$$\begin{aligned} \Upsilon_{r,k}(t) = \frac{2^{\lambda_r-2}\Delta_n(t)}{\sin^2\frac{1}{2}(t-t_r)} \Bigg\{ & \frac{1}{(\lambda_r-2)!}\left(\phi^{\lambda_r-2}_{r,k} - \sum_{\ell=k}^{\lambda_r-3} \phi^\ell_{r,k} D^{\lambda_r-2} h_{\ell,\lambda_r-3}(0)\right) \\ & + \sum_{s=k}^{\lambda_r-3} \frac{(-1)^{\lambda_r-1-s}\phi^s_{r,k}}{2s!} \sum_{\ell=0}^{\left[\frac{1}{2}(\lambda_r-3-s)\right]} \frac{a_{2\ell}(\lambda_r-3)}{(\lambda_r-3-s-2\ell)!} \\ & \times D^{\lambda_r-3-s-2\ell} \cot\frac{1}{2}(t-t_r)\Bigg\}. \end{aligned} \tag{3.35}$$

Theorem 3.2. *Let*

$$T(t) = T\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}\{d_{r,k}\}^{n,\ \lambda_j-1}_{1,\ \ \ 0}(t) = \sum_{r=1}^{n}\sum_{k=0}^{\lambda_r-1} d_{r,k}T_{r,k}(t) \tag{3.36}$$

where $T_{r,k}$ is given by (3.34) *with* (3.35). *Then, T is the unique solution in $H^T_{\frac{1}{2}\lambda}(\alpha)$ of Problem PTHI when $\lambda = \sum_{j=1}^{n} \lambda_j$ is odd, T is the unique solution $H^T_{\frac{1}{2}\lambda}(\alpha)$ of Problem THI when $\lambda = \sum_{j=1}^{n} \lambda_j$ is even. In concrete terms, if T is of degree less than $\frac{1}{2}\lambda$ then both Problems PTHI and THI have only this solution, if T is of degree $\frac{1}{2}\lambda$ then they have a solution in each class $H^T_{\frac{1}{2}\lambda}(\alpha)$ $(\alpha \neq \phi)$, so infinite number of solutions.*

Remark 3.5. We point out the fact that T given by (3.36) depends on the choice of α. While we take $\alpha = [\frac{1}{2}\pi + \phi]_\pi$, then T is called of normal form.

Proof of Theorem 3.2. By (3.28) and (3.33), it is obvious that $T \in H^T_{\frac{1}{2}\lambda}(\alpha)$ and it satisfies the interpolation condition (3.1). For fixed $\alpha \neq \phi$, we prove the uniqueness of T in $H^T_{\frac{1}{2}\lambda}(\alpha)$. In fact, if $G = G\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} \{d_{r,k}\}_{1,\ \ 0}^{n,\ \lambda_j - 1}$ is another one, then we immediately obtain $D^\ell T(t_j) - D^\ell G(t_j) = 0$ $(j = 1, 2, \ldots, n,\ \ell = 0, 1, \ldots, \lambda_j - 1)$, so $T - G = C\Delta_n$ where C is a constant, but $\alpha \neq \phi$, thus $C = 0$. Moreover, if T is of degree less than $\frac{1}{2}\lambda$, then T is of normal form, so the problem discussed has only this solution. If T is of degree $\frac{1}{2}\lambda$, then for the problem discussed there is a solution in each class $H^T_{\frac{1}{2}\lambda}(\alpha)$ $(\alpha \neq \phi)$, so they are different to each other, i.e., the problem discussed has infinite number of solutions. □

Example 3.2. Taking $t_j = \frac{j}{n}\pi$ and $\lambda_j = p+1$ $(j = 0, 1, \ldots, 2n-1)$, then $\lambda = 2n(p+1)$ and $\Delta_{2n}(t) = \frac{1}{2^{(p+1)(2n-1)}} \sin^{p+1} nt$. Obviously, (3.24) becomes

$$\phi = \begin{cases} 0 & \text{if } p \text{ is even,} \\ \frac{1}{2}\pi & \text{if } p \text{ is odd.} \end{cases}$$

In any case we know that Problem THI has the solution T of normal form, more precisely, $T \in H^T_{(p+1)n}(\frac{1}{2}\pi)$ if p is even and $T \in H^T_{(p+1)n}(0)$ if p is odd. This is just the results in [7].

Example 3.3. Let us give a scheme of π-translation nodes. Taking $0 \leqslant t_1 < t_2 < \cdots < t_n < \pi$, $t_{n+j} = \pi + t_j$, $\lambda_{n+j} = \lambda_j$, $d_{n+j,\ell} = (-1)^{\lambda-1} d_{j,\ell}$ $(j = 1, 2, \ldots, n,\ \ell = 0, 1, \ldots, \lambda_j - 1)$ where $\lambda = \sum_{j=1}^{n} \lambda_j$. The Hermite trigonometric interpolation polynomial of normal form is denoted by $T(x) = T\begin{bmatrix} t_1, & t_2, & \ldots, & t_{2n} \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_{2n} \end{bmatrix} \{d_{r,k}\}_{1,\ \ 0}^{2n,\ \lambda_j - 1}(x) \in H^T_\lambda(\alpha)$, then $(-1)^{\lambda-1}T(\pi+x)$ is also one, therefore $T(x) = (-1)^{\lambda-1}T(\pi + x)$. This is to say that T is π-periodic and π-antiperiodic, respectively, when λ is odd and even. Thus, the coefficient of term of degree λ vanishes. So, $T \in H^T_{\lambda-1}$ and is the unique solution of Problem THI under the scheme of π-translation nodes. We get again the result in [1].

4. Hermite interpolation of functions in $C_{2\pi}$ and $\overline{C}_{2\pi}$

Suppose that 2π-antiperiodic function f has derivatives of order up to $\lambda_r - 1$ at the node t_r $(r = 1, 2, \ldots, n)$. We introduce the Hermite paratrigonometric interpolation operator (PTIO) of the form

$$T\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} f(t) = \sum_{r=1}^{n} \sum_{k=0}^{\lambda_r - 1} f^{(k)}(t_r) T_{r,k}(t), \tag{4.1}$$

where $T_{r,k}$' s are given by (3.19) and (3.34) with (3.35), respectively, when $\lambda = \sum_{j=1}^{n} \lambda_j$ is even and odd.

Obviously,

$$T\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} f \in \begin{cases} H^T_{\frac{1}{2}(\lambda-1)} & \text{if } \lambda \text{ is even,} \\ H^T_{\frac{1}{2}\lambda}(\alpha) & \text{if } \lambda \text{ is odd.} \end{cases} \tag{4.2}$$

Let

$$\delta\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} = \mathbf{I} - T\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}, \tag{4.3}$$

where $\mathbf{I}$ is the identity operator. Then $\delta\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}$ is called the remainder of PTIO (4.1). By Theorems 3.1 and 3.2 we have

Lemma 4.1.

$$\ker\left\{\delta\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix}\right\} = \begin{cases} H^T_{\frac{1}{2}(\lambda-1)} & \textit{if } \lambda \textit{ is even}, \\ H^T_{\frac{1}{2}\lambda}(\alpha) & \textit{if} \lambda \textit{ is odd}. \end{cases}$$

When f possesses certain analyticity, we may give the remainder $\delta\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} f$ a clear representation by using the same method of [5]. Assume that f is 2π-antiperiodic function analytic on the rectangular domain $D_r = \{z,\ 0 \leqslant \operatorname{Re} z \leqslant 2\pi, |\operatorname{Im} z| \leqslant r\}$ $(r > 0)$ with the boundary ∂D_r. We denote $f \in A\overline{P}(D_r)$. If f is 2π-periodic function analytic on D_r, we write $f \in AP(D_r)$.

Remark 4.1. In fact, if $f \in A\overline{P}(D_r)$ $(AP(D_r))$, then f is also analytic on the strip region $S_r = \{z,\ |\operatorname{Im} z| < r\}$. To emphasis on this fact we shall use sometimes the denotation $f \in A\overline{P}(S_r)(AP(S_r))$.

In [5], we have proved the following lemma.

Lemma 4.2. *For* $f \in H_n^T(\beta)$ $(n > 0)$, *let*

$$f^*(\tau, t) = \begin{cases} [f(\tau) - f(t)] \cot \frac{1}{2}(\tau - t) & \text{if } \tau \neq t, \\ 2f'(t) & \text{if } \tau = t. \end{cases} \tag{4.4}$$

Sometimes we treat t as a parameter and write $f^*(\tau, t)$ *as* $f_t^*(\tau)$. *Then* $f_t^* \in H_n^T\left([\frac{1}{2}\pi + \beta]_\pi\right)$, *or more precisely,*

$$\begin{aligned} f^*(\tau, t) = {} & a_n \left[\cos(n\tau + \beta) + \cos(nt + \beta)\right] \\ & + \sum_{j=1}^{n-1} \left[A_{n-j}(t) \sin j\tau + B_{n-j}(t) \cos j\tau\right], \end{aligned} \tag{4.5}$$

where $A_j, B_j \in H_j^T$ *and* a_n *is the coefficient of the term of degree n of f.*

Similarly, we have also the following lemma.

Lemma 4.3. *For* $f \in H_n^T$, *let*

$$f_t^\#(\tau) \equiv f^\#(\tau, t) = \begin{cases} [f(\tau) - f(t)] \csc \frac{1}{2}(\tau - t) & \text{if } \tau \neq t, \\ 2f'(t) & \text{if } \tau = t. \end{cases} \tag{4.6}$$

Then $f_t^\# \in H^T_{n-\frac{1}{2}}$. *More precisely,*

$$f^\#(\tau, t) = \sum_{j=0}^{n-1} \left[A_{j+\frac{1}{2}}(t) \cos(n - j - \tfrac{1}{2})\tau + B_{j+\frac{1}{2}}(t) \sin(n - j - \tfrac{1}{2})\tau\right]. \tag{4.7}$$

where $A_{j+\frac{1}{2}}, B_{j+\frac{1}{2}} \in H^T_{j+\frac{1}{2}}$.

Proof. It is sufficient to prove the case $f(t) = \sin(nt + \beta)$ $(n = 1, 2, \ldots)$.

$$\begin{aligned} & [\sin(n\tau + \beta) - \sin(nt + \beta)] \csc \tfrac{1}{2}(\tau - t) \\ & = e^{i\beta} \left[e^{in\tau} - e^{int}\right] \frac{e^{\frac{1}{2}i(\tau+t)}}{e^{i\tau} - e^{it}} + e^{-i\beta} \left[e^{-in\tau} - e^{-int}\right] \frac{e^{-\frac{1}{2}i(\tau+t)}}{e^{-i\tau} - e^{-it}} \\ & = e^{i\beta} \sum_{j=0}^{n-1} e^{i(j+\frac{1}{2})t} e^{i(n-j-\frac{1}{2})\tau} + e^{-i\beta} \sum_{j=0}^{n-1} e^{-i(j+\frac{1}{2})t} e^{-i(n-j-\frac{1}{2})\tau} \\ & = 2 \sum_{j=0}^{n-1} \cos\left((n - j - \tfrac{1}{2})\tau + (j + \tfrac{1}{2})t + \beta\right) \end{aligned}$$

$$= 2\sum_{j=0}^{n-1}\Big[\cos((j+\tfrac{1}{2})t+\beta)\cos(n-j-\tfrac{1}{2})\tau$$

$$- \sin((j+\tfrac{1}{2})t+\beta)\sin(n-j-\tfrac{1}{2})\tau\Big]. \quad \square$$

Introduce the following function:

$$\Lambda_n(\tau,t) = \begin{cases} \big[\Delta_n(\tau)-\Delta_n(t)\big]\csc\tfrac{1}{2}(\tau-t) & \text{if } \lambda \text{ is even,} \\ \Big[\Delta_n(\tau)\frac{\sin(\frac{1}{2}(\tau-t)+\alpha-\phi)}{\sin(\alpha-\phi)}-\Delta_n(t)\Big]\csc\tfrac{1}{2}(\tau-t) & \text{if } \lambda \text{ is odd,} \end{cases} \tag{4.8}$$

where $\alpha \neq \phi$, $\lambda = \sum_{j=1}^{n}\lambda_j$ and ϕ is given by (3.24).

Lemma 4.4.

$$\Lambda_n(\tau,t) = \begin{cases} \sum_{j=0}^{\frac{1}{2}\lambda-1}\Big[A_{j+\frac{1}{2}}(t)\cos(\tfrac{1}{2}\lambda-j-\tfrac{1}{2})\tau \\ \quad +B_{j+\frac{1}{2}}(t)\sin(\tfrac{1}{2}\lambda-j-\tfrac{1}{2})\tau\Big] & \textit{if } \lambda \textit{ is even}, \\ B_0\sin(\tfrac{1}{2}\lambda\tau+\alpha)+\sum_{j=1}^{\frac{1}{2}(\lambda-1)}\big[A_j(t)\cos(\tfrac{1}{2}\lambda-j)\tau \\ \quad +B_j(t)\sin(\tfrac{1}{2}\lambda-j)\tau\big] & \textit{if } \lambda \textit{ is odd}, \end{cases} \tag{4.9}$$

where $A_{j+\frac{1}{2}}, B_{j+\frac{1}{2}} \in H^T_{j+\frac{1}{2}}$, $A_j, B_j \in H^T_j$.

Proof. The first conclusion follows immediately from (4.7). When $\lambda = 1$, (4.9) is clearly true. For $\lambda = 2k+1$ $(k > 0)$, let $p_k(\tau) = \sin^{\lambda_1-1}\frac{1}{2}(\tau-t_1)\prod_{j=2}^{n}\sin^{\lambda_j}\frac{1}{2}(\tau-t_j)$. Then, by Lemma 2.1,

$$\begin{aligned} p_k(t) &= B_0^1\sin(kt+\beta)+q_{k-1}(t), \quad B_0^1 \neq 0, \\ \beta &= \left[\frac{1}{2}(\pi+t_1)+\phi\right]_\pi, \quad q_{k-1}\in H^T_{k-1} \end{aligned} \tag{4.10}$$

and in concrete terms,

$$p_k(t) = \frac{(-1)^k}{2^{2k-1}}\sin(kt+\tfrac{1}{2}(\pi+t_1)+\phi)+q_{k-1}(t). \tag{4.11}$$

By using (4.5), we have

$$
\begin{aligned}
&\Lambda_n(\tau,t)\\
&=\left[\Delta_n(\tau)\cos\frac{1}{2}(\tau-t)-\Delta_n(t)\right]\csc\frac{1}{2}(\tau-t)+\cot(\alpha-\phi)\Delta_n(\tau)\\
&=\sin\frac{1}{2}(\tau-t_1)\,[p_k(\tau)-p_k(t)]\cot\frac{1}{2}(\tau-t)+\cos\frac{1}{2}(\tau-t_1)\,p_k(t)\\
&\quad+\cot(\alpha-\phi)\sin\frac{1}{2}(\tau-t_1)\left[B_0^1\sin(k\tau+\beta)+q_{k-1}(\tau)\right]\\
&=\frac{B_0^1}{\sin(\alpha-\phi)}\sin\tfrac{1}{2}(\tau-t_1)\left[\sin(\alpha-\phi)\cos(k\tau+\beta)+\cos(\alpha-\phi)\sin\tfrac{1}{2}(k\tau+\beta)\right]\\
&\quad+\sin\tfrac{1}{2}(\tau-t_1)\sum_{j=1}^{k-1}\left[A_j^1(t)\sin(k-j)\tau+B_j^1(t)\cos(k-j)\tau\right]\\
&\quad+B_0^1\sin\frac{1}{2}(\tau-t_1)\cos(kt+\beta)+\cos\frac{1}{2}(\tau-t_1)\,p_k(t)\\
&\quad+\cot(\alpha-\phi)\sin\frac{1}{2}(\tau-t_1)q_{k-1}(\tau)\\
&=B_0\sin\left(\frac{1}{2}\lambda\tau+\alpha\right)+\sum_{j=1}^{\frac{1}{2}(\lambda-1)}\left[A_j(t)\cos\left(\frac{1}{2}\lambda-j\right)\tau+B_j(t)\sin\left(\frac{1}{2}\lambda-j\right)\tau\right],
\end{aligned}
$$

where $A_j^1, B_j^1 \in H_j^T$, A_j and B_j are some linear associative forms of A_ℓ^1 and B_ℓ^1 $(\ell \leqslant j)$. So, we know $A_j, B_j \in H_j^T$. Thus, the second conclusion is proved. □

Lemma 4.5. *If* $f \in A\overline{P}(D_r)$ *then*

$$
f(\tau)=\frac{1}{4\pi i}\int_{\partial D_r} f(z)\csc\frac{1}{2}(z-\tau)\,\mathrm{d}z,\quad \tau\in S_r, \tag{4.12}
$$

where the above integral is understood as the Cauchy principle value integral if $\tau \equiv iy$ $(\bmod 2\pi)$ *with real* y $(|y|<r)$.

Proof. Let us denote the interior of D_r by D_r^0. By the residue theorem (4.12) is true for $\tau \in D_r^0$. If $\tau = iy$, by the extended residue theorem due to Jian-ke Lu [8, p. 75], we get

$$
\begin{aligned}
\frac{1}{4\pi i}\int_{\partial D_r} f(z)\csc\frac{1}{2}(z-\tau)\,\mathrm{d}z &= \frac{1}{2}[\mathrm{sp}(iy)\mathrm{res}(iy)+\mathrm{sp}(2\pi+iy)\mathrm{res}(2\pi+iy)]\\
&=\frac{1}{2}[f(iy)-f(2\pi+iy)]=f(iy),
\end{aligned}
$$

where $\mathrm{sp}(x)$ denotes the span at x with respect to ∂D_r (for example, $\mathrm{sp}(iy) = \mathrm{sp}(2\pi + iy) = \frac{1}{2}$ for $|y| < r$) and $\mathrm{res}(x)$ is the residue of integrand $f(z) \csc \frac{1}{2}(z - \tau)$ at x. Finally, noting that both the function on the left-hand side and one on the right-hand side in (4.12) are 2π-antiperiodic, the proof is completed. □

Let

$$\Theta_n(\tau) = \frac{1}{4\pi i} \int_{\partial D_r} f(z) \frac{\Delta_n(\tau, z)}{\Delta_n(z)}\, \mathrm{d}z. \tag{4.13}$$

In (4.13), if $\Delta_n(0) = \Delta_n(2\pi) = 0$, say $t_1 = 0$, then we understand it as the Cauchy principal value integral when $\lambda_1 = 1$ and the singular integral of higher order when $\lambda_1 > 1$. Thus, by (4.9) we get

$$\Theta_n \in \begin{cases} H^T_{\frac{1}{2}(\lambda-1)} & \text{if } \lambda \text{ is even,} \\ H^T_{\frac{1}{2}\lambda}(\alpha) & \text{if } \lambda \text{ is odd.} \end{cases} \tag{4.14}$$

Now we prove the following approximation theorem.

Theorem 4.1. *If $f \in A\overline{P}(D_r)$, then*

$$T \begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} f(\tau) = \frac{1}{4\pi i} \int_{\partial D_r} f(z) \frac{\Delta_n(\tau, z)}{\Delta_n(z)}\, \mathrm{d}z \tag{4.15}$$

$$\delta \begin{bmatrix} t_1, t_2, \ldots, t_n \\ \lambda_1, \lambda_2, \ldots, \lambda_n \end{bmatrix} f(\tau) = \begin{cases} \frac{\Delta_n(\tau)}{4\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\, \mathrm{d}z & \textit{if } \lambda \textit{ is even}, \\ \frac{\Delta_n(\tau)}{4\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \left[\cot \frac{1}{2}(z-\tau) - \cot \frac{1}{2}(\alpha-\phi)\right] \mathrm{d}z & \textit{if } \lambda \textit{ is odd}, \end{cases} \tag{4.16}$$

or

$$\delta \begin{bmatrix} t_1, \ldots, t_n \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} f(\tau) = \begin{cases} \frac{1}{2\pi} \mathrm{Re}\left\{ i\Delta_n(\tau) \int_{ir}^{2\pi+ir} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\, \mathrm{d}z \right\} & \textit{if } \lambda \textit{ is even}, \\ \frac{1}{2\pi} \mathrm{Re}\left\{ i\Delta_n(\tau) \int_{ir}^{2\pi+ir} \frac{f(z)}{\Delta_n(z)} \left[\cot \frac{1}{2}(z-\tau) - \cot \frac{1}{2}(\alpha-\phi)\right] \mathrm{d}z \right\} & \textit{if } \lambda \textit{ is odd} \end{cases} \tag{4.17}$$

and

$$\left\| \delta \begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} f \right\| \leqslant \coth\left(\frac{r}{2}\right) \|f\|_r \, \|\Delta_n\| \left\| (\Delta_n)^{-1} \right\|_r, \tag{4.18}$$

where (4.16) *is the singular integral of higher order if* $\Delta_n(0) = 0$ *or* $\tau = 0$, $\|\cdot\|$ *and* $\|\cdot\|_r$ *denotes the sup-norm of a function on* $[0, 2\pi]$ *and on the line-segment* $z = x + ir$ $(0 \leqslant x \leqslant 2\pi)$, *respectively*.

Obviously, by (4.12) and (4.13), we have, for $\tau \in S_r$,

$$f(\tau) - \Theta_n(\tau) = \begin{cases} \frac{\Delta_n(\tau)}{4\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z & \text{if } \lambda \text{ is even,} \\ \frac{\Delta_n(\tau)}{4\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \\ \quad \times \left[\cot \frac{1}{2}(z-\tau) - \cot \frac{1}{2}(\alpha-\phi)\right] \mathrm{d}z & \text{if } \lambda \text{ is odd.} \end{cases} \tag{4.19}$$

If we demonstrate the equalities

$$f^{\ell}(t_j) = \Theta_n^{\ell}(t_j), \quad j = 1, 2, \ldots, n, \quad \ell = 0, 1, \ldots, \lambda_j - 1, \tag{4.20}$$

then, by (4.14) and Lemma 4.1, (4.15) and (4.16) would follow. We may get directly $f(t_j) = \Theta_n(t_j)$ $(j = 1, 2, \ldots, n)$ from (4.19), but to get whole (4.20) is not too easy. To do so, we prove the following stronger result.

Lemma 4.6. *For* $f \in A\overline{P}(S_r)$, *let*

$$F_n(\tau) = \begin{cases} \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z & \textit{if } \lambda \textit{ is even,} \\ \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \cot \frac{1}{2}(z-\tau)\,\mathrm{d}z & \textit{if } \lambda \textit{ is odd.} \end{cases} \tag{4.21}$$

Then

$$F_n \in \begin{cases} A\overline{P}(S_r) & \textit{if } \lambda \textit{ is even,} \\ AP(S_r) & \textit{if } \lambda \textit{ is odd.} \end{cases} \tag{4.22}$$

Proof. We prove this lemma only for the case of even λ, the case of odd λ is similar. Firstly, F_n is obviously 2π-antiperiodic. Secondly, if $\tau \in S_r$ but $\tau \not\equiv t_j \pmod{2\pi}$ $(j = 1, 2, \ldots, t_n)$, then F_n is analytic at τ by (4.19). Thus, we only need to prove that F_n is analytic at t_j $(j = 1, 2, \ldots, t_n)$. For simplicity, we write $0 \leqslant t_1 < t_2 < \cdots < t_n < 2\pi$.

Case I: When $\Delta_n(0) \neq 0$, so $\Delta_n(z) \neq 0$ for $z \in \partial D_r$. Thus, the proof is very simple, since F_n is analytic in D_r^0 [10] and t_j $(j = 1, 2, \ldots, t_n)$ is just in D_r^0.

Case II: When $\Delta_n(0) = 0$, so $t_1 = 0 \in \partial D_r$. Thus, the proof is more complicated, since F_n is a singular integral of higher order.

In this case, let

$$g_n(z, \tau) = \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z - \tau), \quad \operatorname{res}(g(\cdot, \tau), t) = \operatorname{res}(\tau, t). \tag{4.23}$$

For $\tau \in S_r$, we partition the calculation of $F_n(\tau)$ into four cases using the residue theorem for singular integrals of higher order (see [8, p. 75]).

(1) When $\tau \neq t_j$ $(j = 1, 2, \ldots, n)$ and is in D_r^0, then

$$F_n(\tau) = \frac{1}{2}[\operatorname{res}(\tau, 0) + \operatorname{res}(\tau, 2\pi)] + \sum_{j=2}^{n} \operatorname{res}(\tau, t_j) + \frac{2f(\tau)}{\Delta_n(\tau)}. \tag{4.24}$$

We point out the important relation

$$\mathrm{res}(\tau, 0) = \mathrm{res}(\tau, 2\pi) \quad \text{for any } \tau \in S_r. \tag{4.25}$$

In fact, if

$$g_n(z, \tau) = \sum_{j=-\lambda_j(\tau)}^{+\infty} a_j(\tau) z^j, \quad |z| < \eta,$$

where $\lambda_j(\tau) = \lambda_j$ when $\tau \neq t_j$ and $\lambda_j(\tau) = \lambda_j + 1$ when $\tau = t_j$, then

$$g_n(z, \tau) = g_n(z - 2\pi, \tau) = \sum_{j=-\lambda_j(\tau)}^{+\infty} a_j(\tau)(z - 2\pi)^j, \quad |z - 2\pi| < \eta.$$

So,

$$F_n(\tau) = \sum_{j=1}^{n} \mathrm{res}(\tau, t_j) + \frac{2f(\tau)}{\Delta_n(\tau)}. \tag{4.26}$$

(2) When $\tau = t_j$, similarly

$$F_n(t_j) = \sum_{\ell=1}^{n} \mathrm{res}(t_j, t_\ell), \quad j = 1, 2, \ldots, n. \tag{4.27}$$

(3) When $\tau = iy$ $(0 < |y| < r)$, then

$$\begin{aligned} F_n(\tau) &= \frac{1}{2}[\mathrm{res}(iy, 0) + \mathrm{res}(iy, 2\pi)] + \sum_{j=2}^{n} \mathrm{res}(iy, t_j) \\ &\quad + \frac{1}{2}[\mathrm{res}(iy, \tau) + \mathrm{res}(iy, 2\pi + \tau)] \\ &= \sum_{j=1}^{n} \mathrm{res}(iy, t_j) + \frac{2f(iy)}{\Delta_n(iy)}. \end{aligned} \tag{4.28}$$

(4) When $\tau \in S_r$, there exists a point τ_0 with $\mathrm{Re}\,\tau_0 \in [0, 2\pi)$ and an integer k such that $\tau = 2k\pi + \tau_0$. Then

$$F_n(\tau) = (-1)^k F_n(\tau_0). \tag{4.29}$$

In words, for any $\tau \in S_r$ we have

$$F_n(\tau) = \begin{cases} (-1)^k \sum_{\ell=1}^{n} \mathrm{res}(t_j, t_\ell) & \text{if } \tau = 2k\pi + t_j \\ & \quad (j = 1, 2, \ldots, n), \\ (-1)^k \left[\sum_{j=1}^{n} \mathrm{res}(\tau_0, t_j) + \frac{2f(\tau_0)}{\Delta_n(\tau_0)} \right] & \text{if } \tau = 2k\pi + \tau_0,\ \tau_0 \neq t_j \\ & \quad (j = 1, 2, \ldots, n). \end{cases} \tag{4.30}$$

Next, we make a new integral for $g(z, \tau)$. Let G_r denote the closed polygon bounded by the polygonal line $\left[2\pi - ir, \frac{1}{2}(t_n + 2\pi), 2\pi + ir, ir, \frac{1}{2}(t_n - 2\pi), -ir\right]$. Set

$$H_n(\tau) = \begin{cases} \frac{1}{2\pi i} \int_{\partial G_r} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z - \tau)\, \mathrm{d}z & \text{if } \lambda \text{ is even}, \\ \frac{1}{2\pi i} \int_{\partial G_r} \frac{f(z)}{\Delta_n(z)} \cot \frac{1}{2}(z - \tau)\, \mathrm{d}z & \text{if } \lambda \text{ is odd}, \end{cases} \tag{4.31}$$

Now we again consider four cases to calculate $H_n(\tau)$. Still assume that λ is even.

(1) Let G_r^0 denote the interior of G_r. If $\tau_0 \in G_r^0$, by the residue theorem we get

$$H_n(\tau_0) = \begin{cases} \sum_{\ell=1}^{n} \mathrm{res}(t_j, t_\ell) & \text{if } \tau_0 = t_j \quad (j = 1, 2, \ldots, n), \\ \sum_{\ell=1}^{n} \mathrm{res}(\tau_0, t_j) + \frac{2f(\tau_0)}{\Delta_n(\tau_0)} & \text{if } \tau_0 \neq t_j,\ \tau_0 \in G_r^0. \end{cases} \tag{4.32}$$

(2) Let $\lhd$ denote the triangle bounded by the polygonal line $\left[2\pi - ir, 2\pi + ir, \frac{1}{2}(t_n + 2\pi)\right]$. If $\tau_0 \in \lhd^0$, by the residue theorem we get

$$\begin{aligned} H_n(\tau_0) &= \sum_{j=1}^{n} \mathrm{res}(\tau_0, t_j) + \mathrm{res}(\tau_0, -2\pi + \tau_0) \\ &= \sum_{j=1}^{n} \mathrm{res}(\tau_0, t_j) - \frac{2f(-2\pi + \tau_0)}{\Delta_n(-2\pi + \tau_0)} \\ &= \sum_{j=1}^{n} \mathrm{res}(\tau_0, t_j) + \frac{2f(\tau_0)}{\Delta_n(\tau_0)}. \end{aligned} \tag{4.33}$$

In passing, here we have proved again that

$$\mathrm{res}(\tau_0, -2\pi + \tau_0) = \mathrm{res}(\tau_0, \tau_0) \quad \text{for any } \tau_0 \neq t_j \ (j = 1, 2, \ldots, n). \tag{4.34}$$

(3) If τ_0 lies on the polygonal line $\left[2\pi - ir, 2\pi + ir, \frac{1}{2}(t_n + 2\pi)\right]$ but $\tau_0 \neq 2\pi - ir, 2\pi + ir$, then, by the extended residue theorem we get

$$H_n(\tau_0) = \sum_{j=1}^{n} \operatorname{res}(\tau_0, t_j) + \big[\operatorname{sp}(-2\pi + \tau_0)\operatorname{res}(\tau_0, -2\pi + \tau_0) + \operatorname{sp}(\tau_0)\operatorname{res}(\tau_0, \tau_0)\big], \tag{4.35}$$

where $\operatorname{sp}(x)$ denotes the span at x with respect to ∂G_r. Noting (4.34) and

$$\begin{cases} \operatorname{sp}(-2\pi + \tau_0) = \operatorname{sp}(\tau_0) = \frac{1}{2} \quad \text{if} \ \ \tau_0 \neq \frac{1}{2}(t_n + 2\pi), \\ \operatorname{sp}\left(\frac{1}{2}(2\pi + t_n)\right) = 1 - \frac{1}{\pi} \arctan \frac{2r}{2\pi - t_n}, \\ \operatorname{sp}\left(\frac{1}{2}(t_n - 2\pi)\right) = \frac{1}{\pi} \arctan \frac{2r}{2\pi - t_n}, \end{cases} \tag{4.36}$$

we finally have

$$H_n(\tau_0) = \sum_{j=1}^{n} \operatorname{res}(\tau_0, t_j) + \frac{2f(\tau_0)}{\Delta_n(\tau_0)}. \tag{4.37}$$

(4) If $\tau \in S_r$ then

$$H_n(\tau) = (-1)^k H_n(\tau_0), \quad \tau = 2k\pi + \tau_0. \tag{4.38}$$

To sum up, we have

$$H_n(\tau) = \begin{cases} (-1)^k \sum_{\ell=1}^{n} \operatorname{res}(t_j, t_\ell) & \text{if} \ \ \tau = 2k\pi + t_j \\ & (j = 1, 2, \ldots, n), \\ (-1)^k \left[\sum_{j=1}^{n} \operatorname{res}(\tau_0, t_j) + \frac{2f(\tau_0)}{\Delta_n(\tau_0)} \right] & \text{if } \tau = 2k\pi + \tau_0, \tau_0 \neq t_j \\ & (j = 1, 2, \ldots, n). \end{cases} \tag{4.39}$$

By (4.30) and (4.39), we obtain

$$F_n(\tau) = H_n(\tau), \quad \tau \in S_r. \tag{4.40}$$

Remark 4.2. The proof for the case λ being odd is similar.

Noting that H_n is analytic on G_r^0, in particular at t_j $(j = 1, 2, \ldots, n)$, and hence so is F_n. Hence we have pointed out above that F_n is analytic at $\tau \neq t_j$, therefore F_n is analytic on S_r. Moreover, H_n is also analytic on S_r. □

Proof of Theorem 4.1. By the analyticity of F_n (4.20) holds, consequently, both (4.15) and (4.16) are true. Noting that H_n is only a Cauchy principle value integral, by the 2π-periodicity of the expressions $\frac{f(z)}{\Delta_n(z)}$ csc $\frac{1}{2}(z-\tau)$ and $\frac{f(z)}{\Delta_n(z)}$ cot $\frac{1}{2}(z-\tau)$ with respect to the variable z, we calculate that

$$H_n(\tau) = \begin{cases} \frac{1}{2\pi i}\left\{\int_{-ir}^{2\pi-ir} - \int_{ir}^{2\pi+ir}\right\} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z & \text{if } \lambda \text{ is even,} \\ \frac{1}{2\pi i}\left\{\int_{-ir}^{2\pi-ir} - \int_{ir}^{2\pi+ir}\right\} \frac{f(z)}{\Delta_n(z)} \cot \frac{1}{2}(z-\tau)\,\mathrm{d}z & \text{if } \lambda \text{ is odd.} \end{cases} \tag{4.41}$$

By (4.40)

$$\delta\begin{bmatrix} t_1, \ldots, t_n \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} f(\tau) = \begin{cases} \frac{\Delta_n(\tau)}{4\pi i}\left\{\int_{-ir}^{2\pi-ir} - \int_{ir}^{2\pi+ir}\right\} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z \\ \qquad\qquad \text{if } \lambda \text{ is even,} \\ \frac{\Delta_n(\tau)}{4\pi i}\left\{\int_{-ir}^{2\pi-ir} - \int_{ir}^{2\pi+ir}\right\} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)}\left[\cot \frac{1}{2}(z-\tau) - \cot \frac{1}{2}(\alpha-\phi)\right]\mathrm{d}z \\ \qquad\qquad \text{if } \lambda \text{ is odd,} \end{cases} \tag{4.42}$$

which holds on S_r. If $\tau \in [0, 2\pi]$, then (4.17) holds. In fact, noting that f possesses the Schwarz symmetry (i.e., $f(\overline{z}) = \overline{f(z)}$) by $f(\mathcal{R}) \subseteq (\mathcal{R})$ ($\mathcal{R}$ denotes the set of real numbers) and the principle of the Schwarz symmetric extension, so do the integrands in (4.42). Thus, (4.17) results from (4.42), (4.18) follows from (4.17). □

Remark 4.3. If we replace ∂D_r in the integrals (4.12) and (4.13) by ∂G_r, then (4.17) and (4.18) might be obtained quickly . But we do not get (4.16), which will play a very important role in the quadrature formulas of singular integral with the cosecant kernel.

Remark 4.4. By (4.41), we see

$$\begin{cases} \int_{-ir}^{ir} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z = \int_{2\pi-ir}^{2\pi+ir} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z & \text{if } \lambda \text{ is even,} \\ \int_{-ir}^{ir} \frac{f(z)}{\Delta_n(z)} \cot \frac{1}{2}(z-\tau)\,\mathrm{d}z = \int_{2\pi-ir}^{2\pi+ir} \frac{f(z)}{\Delta_n(z)} \cot \frac{1}{2}(z-\tau)\,\mathrm{d}z & \text{if } \lambda \text{ is odd.} \end{cases} \tag{4.43}$$

This fact is very interesting, but not obvious since these integrals are singular integrals of higher order when $t_1 = 0$.

Corollary 4.1. *If* $f \in A\overline{P}(D_r)$, *then*

$$\left\| \delta \begin{bmatrix} t_1, \dots, t_n \\ \lambda_1, \dots, \lambda_n \end{bmatrix} f \right\| \leqslant \coth \frac{r}{2} \, \|f\|_r \, \sinh^{-n} \frac{r}{2}, \tag{4.44}$$

In particular, if $r > 2\mathrm{arcsinh}1 = 2\ln(1+\sqrt{2})$, *then* $\lim_{\lambda\to\infty} \left\| \delta \begin{bmatrix} t_1, \dots, t_n \\ \lambda_1, \dots, \lambda_n \end{bmatrix} f \right\| = 0$ *where* $\lambda = \sum_{j=1}^{n} \lambda_j$.

Suppose that 2π-periodic function f has derivatives of order up to $\lambda_r - 1$ at the node t_r $(r = 1, 2, \dots, n)$. We introduce the Hermite trigonometric interpolation operator (TIO) of the form

$$T \begin{bmatrix} t_1, & t_2, & \dots, & t_n \\ \lambda_1, & \lambda_2, & \dots, & \lambda_n \end{bmatrix} f(t) = \sum_{r=1}^{n} \sum_{k=0}^{\lambda_r - 1} f^{(k)}(t_r) T_{r,k}(t), \tag{4.45}$$

where $T_{r,k}$' s are given by (3.19) and (3.34) with (3.35), respectively, when $\lambda = \sum_{j=1}^{n} \lambda_j$ is odd and even.

Obviously,

$$T \begin{bmatrix} t_1, & t_2, & \dots, & t_n \\ \lambda_1, & \lambda_2, & \dots, & \lambda_n \end{bmatrix} f \in \begin{cases} H^T_{\frac{1}{2}(\lambda-1)} & \text{if } \lambda \text{ is odd,} \\ H^T_{\frac{1}{2}\lambda}(\alpha) & \text{if } \lambda \text{ is even.} \end{cases} \tag{4.46}$$

Denote the remainder as

$$\delta \begin{bmatrix} t_1, & t_2, & \dots, & t_n \\ \lambda_1, & \lambda_2, & \dots, & \lambda_n \end{bmatrix} = \mathbf{I} - T \begin{bmatrix} t_1, & t_2, & \dots, & t_n \\ \lambda_1, & \lambda_2, & \dots, & \lambda_n \end{bmatrix}. \tag{4.47}$$

By Theorems 3.1 and 3.2 we have also

Lemma 4.7.

$$\ker \left\{ \delta \begin{bmatrix} t_1, & t_2, & \dots, & t_n \\ \lambda_1, & \lambda_2, & \dots, & \lambda_n \end{bmatrix} \right\} = \begin{cases} H^T_{\frac{1}{2}(\lambda-1)} & \textit{if } \lambda \textit{ is odd,} \\ H^T_{\frac{1}{2}\lambda}(\alpha) & \textit{if } \lambda \textit{ is even.} \end{cases}$$

Now introduce the following function:

$$\wedge_n(\tau, t) = \begin{cases} \left[\Delta_n(\tau) - \Delta_n(t) \cos \frac{1}{2} (\tau - t)\right] \csc \frac{1}{2} (\tau - t) & \text{if } \lambda \text{ is odd,} \\ \left[\Delta_n(\tau) \dfrac{\sin\left(\frac{1}{2}(\tau - t) + \alpha - \phi\right)}{\sin(\alpha - \phi)} - \Delta_n(t) \cos \frac{1}{2} (\tau - t) \right] & \\ \times \csc \frac{1}{2} (\tau - t) & \text{if } \lambda \text{ is even,} \end{cases} \tag{4.48}$$

where $\alpha \neq \phi$, $\lambda = \sum_{j=1}^{n} \lambda_j$ and ϕ is given by (3.24).

In exactly the same way, we have the following lemmas which are completely parallel to Lemmas 4.4–4.6.

Lemma 4.8.

$$\Lambda_n(\tau,t)=\begin{cases}\sum_{j=0}^{\frac{1}{2}(\lambda-1)}\left[A_{\frac{1}{2}\lambda-j}(t)\cos j\tau + B_{\frac{1}{2}\lambda-j}(t)\sin j\tau\right] & \textit{if } \lambda \textit{ is odd},\\ B_0\sin(\frac{1}{2}\lambda\tau+\alpha) & \\ \quad +\sum_{j=0}^{\frac{1}{2}\lambda-1}\left[A_{\frac{1}{2}\lambda-j}(t)\cos j\tau+B_{\frac{1}{2}\lambda-j}(t)\sin j\tau\right] & \textit{if } \lambda \textit{ is even},\end{cases} \tag{4.49}$$

where $A_{j+\frac{1}{2}}, B_{j+\frac{1}{2}} \in H^T_{j+\frac{1}{2}}$, $A_j, B_j \in H^T_j$.

Lemma 4.9. *If* $f \in AP(D_r)$ *then*

$$f(\tau) = \frac{1}{4\pi i}\int_{\partial D_r} f(z)\cot\frac{1}{2}(z-\tau)\,\mathrm{d}z, \quad \tau \in S_r, \tag{4.50}$$

where the above integral is understood as the Cauchy principle value integral if $\tau \equiv iy \pmod{2\pi}$ *with real* y $(|y| < r)$.

Lemma 4.10. *If* $f \in AP(S_r)$, *let*

$$E_n(\tau) = \begin{cases}\frac{1}{2\pi i}\int_{\partial D_r}\frac{f(z)}{\Delta_n(z)}\csc\frac{1}{2}(z-\tau)\,\mathrm{d}z & \textit{if } \lambda \textit{ is odd},\\ \frac{1}{2\pi i}\int_{\partial D_r}\frac{f(z)}{\Delta_n(z)}\cot\frac{1}{2}(z-\tau)\,\mathrm{d}z & \textit{if } \lambda \textit{ is even},\end{cases} \tag{4.51}$$

then

$$E_n \in \begin{cases} A\overline{P}(S_r) & \textit{if } \lambda \textit{ is odd},\\ AP(S_r) & \textit{if } \lambda \textit{ is even}.\end{cases} \tag{4.52}$$

By these lemmas, we get the following result in an obvious manner similar to that used before.

Theorem 4.2. *If* $f \in AP(D_r)$, *then*

$$T\begin{bmatrix} t_1, & t_2, & \dots, & t_n \\ \lambda_1, & \lambda_2, & \dots, & \lambda_n\end{bmatrix} f(\tau) = \frac{1}{4\pi i}\int_{\partial D_r} f(z)\frac{\wedge_n(\tau,z)}{\Delta_n(z)}\,\mathrm{d}z, \tag{4.53}$$

$$\delta\begin{bmatrix} t_1, t_2, \ldots, t_n \\ \lambda_1, \lambda_2, \ldots, \lambda_n \end{bmatrix} f(\tau) = \begin{cases} \frac{\Delta_n(\tau)}{4\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z & \text{if } \lambda \text{ is odd}, \\ \frac{\Delta_n(\tau)}{4\pi i} \int_{\partial D_r} \frac{f(z)}{\Delta_n(z)} \Big[\cot \frac{1}{2}(z-\tau) \\ \quad - \cot \frac{1}{2}(\alpha-\phi)\Big]\,\mathrm{d}z & \text{if } \lambda \text{ is even}, \end{cases} \tag{4.54}$$

or

$$\delta\begin{bmatrix} t_1, \ldots, t_n \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} f(\tau) = \begin{cases} \frac{1}{2\pi} \mathrm{Re}\Big\{ i\,\Delta_n(\tau) \int_{ir}^{2\pi+ir} \frac{f(z)}{\Delta_n(z)} \csc \frac{1}{2}(z-\tau)\,\mathrm{d}z \Big\} & \text{if } \lambda \text{ is odd}, \\ \frac{1}{2\pi} \mathrm{Re}\Big\{ i\,\Delta_n(\tau) \int_{ir}^{2\pi+ir} \frac{f(z)}{\Delta_n(z)} \Big[\cot \frac{1}{2}(z-\tau) \\ \quad -\cot \frac{1}{2}(\alpha-\phi)\Big]\,\mathrm{d}z \Big\} & \text{if } \lambda \text{ is even}, \end{cases} \tag{4.55}$$

and

$$\left\| \delta\begin{bmatrix} t_1, & t_2, & \ldots, & t_n \\ \lambda_1, & \lambda_2, & \ldots, & \lambda_n \end{bmatrix} f \right\| \leqslant \coth\left(\frac{r}{2}\right) \|f\|_r \, \|\Delta_n\| \, \left\| (\Delta_n)^{-1} \right\|_r . \tag{4.56}$$

Example 4.1. As in Example 3.2 we take $\Delta_{2n}(t) = 2^{-(p+1)(2n-1)} \sin^{p+1}(nt+\theta)$ with an arbitrary real number θ. From (4.56) we get

$$\left\| \delta\begin{bmatrix} t_1, \ldots, t_n \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} f \right\| = O\left(e^{-\frac{1}{2}\lambda r}\right) \quad \text{as} \quad \lambda \to +\infty, \tag{4.57}$$

where $\lambda = 2n(p+1)$.

For the more general case, we have the following.

Corollary 4.2. *If* $f \in AP(D_r)$, *then*

$$\left\| \delta\begin{bmatrix} t_1, \ldots, t_n \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} f \right\| \leqslant \coth \frac{r}{2} \, \|f\|_r \, \sinh^{-n} \frac{r}{2}. \tag{4.58}$$

In particular, if $r > 2\operatorname{arcsinh}1 = 2\ln(1+\sqrt{2})$, *then* $\lim_{\lambda\to\infty} \left\| \delta\begin{bmatrix} t_1, \ldots, t_n \\ \lambda_1, \ldots, \lambda_n \end{bmatrix} f \right\| = 0$ *where* $\lambda = \sum_{j=1}^{n} \lambda_j$.

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